Orbit Determination for a Thrusted Space Vehicle

GILBERT C. CARPENTER*
Grumman Aircraft Engineering Corporation, Bethpage, N. Y.

AND

EDWARD T. PITKIN†
University of Connecticut, Storrs, Connecticut

This paper is concerned with the problem of determining the time history of position, velocity and thrust acceleration of a space vehicle given only angular observations at discrete times and without prior knowledge of a nominal trajectory. It is shown that the motion, which involves unmeasurable state variables and partially unknown system dynamics, can be mathematically modeled by a system of ordinary differential equations subject to distributed boundary conditions. The resulting system is then solved by iteration with a modified quasilinearization technique. Results obtained from application of the method to a logarithmic spiral orbit will be examined. It will be shown that the method exhibits high precision and fast convergence even with poor estimates of the initial values for the state vector that is used to start the iterative process.

1. Introduction

CONTINUOUSLY thrusted interplanetary vehicles of the future will require new techniques for their guidance and navigation. At the present time much research has been done on trajectory analysis and optimal control for these missions, yet little work has been done on the critical problem of orbit determination. Jordan[1] has shown that it is possible to solve the problem of determining small perturbations in the vehicle's position, velocity, and acceleration by applying filtering techniques to the data if one has prior knowledge of the vehicle's nominal trajectory. Apparently no one, however, has solved the more difficult problem of determining the completely unknown position, velocity, and thrust program of a powered vehicle from the types of observations which are now available; namely telescopic (angular observations) and radar (range and range-rate), without knowledge of a good approximate nominal.

In this paper it will be shown that the motion of a thrusted vehicle, which involves unmeasurable state variables and partially unknown system dynamics can be mathematically modeled by a system of ordinary differential equations subject to multipoint boundary conditions. Prior knowledge of the vehicle's nominal trajectory is not necessary for a solution. Furthermore, the method of solution is not necessarily restricted to the class of problems where the thrust acceleration magnitude is small compared to the gravitational acceleration. The analytic method of differential approximation and the numerical technique of quasilinearization are the basic mathematical tools which will be used to solve the problem. First, the unknown portion of the system dynamics, that which describes the thrust program, is represented by the method of differential approximation which was originally introduced by Bellman[2] for the purpose of state estimation, and extended to the present application by Carpenter.[3] Upon application of differential approximation the complete system dynamics can be described by a set of ordinary differential equations. The technique of quasilinearization as modified by Pitkin and Carpenter[4] is then used to numerically solve the resulting multipoint boundary value problem.

First we shall summarize the equations of motion and observation; restricting ourselves in this paper to the case of angular observations for simplicity. (Range and range-rate data can be incorporated with additional algebraic complexity.) We will then show how the system is converted via differential approximation to a multipoint boundary value problem and discuss its solution by quasilinearization. Finally we shall present results for the two particular orbit determination problems, one orbit generated by sinusoidal thrust variation and the other a logarithmic spiral. It will be shown that the method exhibits high precision and fast convergence even with poor estimates of the initial values of the components of the state vector used to start the iterative process.

2. Equations of Motion and Observation

The motion of a vehicle in an inverse square gravitational field subject to an additional acceleration caused by thrust may be expressed via the differential equation

$$\ddot{\mathbf{r}} = -\mu\mathbf{r}/r^3 + \mathbf{u} \tag{1}$$

which may also be expressed in matrix form as

$$\dot{\mathbf{y}} = \begin{pmatrix}\dot{\mathbf{r}}\\ \dot{\mathbf{v}}\end{pmatrix} = \begin{pmatrix}0 & I_3\\ \dfrac{-\mu I_3}{\mathbf{r}^3} & 0\end{pmatrix}\begin{pmatrix}\mathbf{r}\\ \mathbf{v}\end{pmatrix} + \begin{pmatrix}0\\ \mathbf{u}\end{pmatrix} \tag{2}$$

where $\mathbf{r}$ is the position vector, $\mathbf{v}$ the velocity and $\mathbf{u}$ the thrust acceleration. We presume that $\mathbf{u}$ is a fairly smooth function of time possessing enough derivatives to be reasonably approximated by the method which is to follow. The vector $\mathbf{y}(t)$ composed of $\mathbf{r}$ and $\mathbf{v}$ is the state of the dynamical system in the six-dimensional state space.

Neither the full state vector nor any of its components are assumed to be observed. Rather the angular data, right ascension, α, and declination, δ, at W discrete times are recorded from one or more observation platforms located at positions not coinciding with the dynamical center. It is evident from Fig. 1 that the position vector is related to these observations through the relation

$$\mathbf{r}(t) + \mathbf{R}(t) = \rho(t)\mathbf{L}(t) \tag{3}$$

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* Research Scientist.

† Professor of Aerospace Engineering. Associate Fellow AIAA.

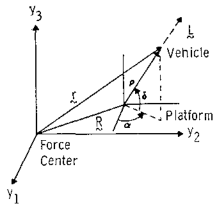

Fig. 1 Observation platform geometry.

in which

$$L_1 = \cos\delta \cos\alpha;\; L_2 = \cos\delta \sin\alpha;\; L_3 = \sin\delta \tag{4}$$

and the position, $\mathbf{R}$, of the dynamical center with respect to the observation platform is presumably known and recorded at the observation times t_i. Our problem may then be stated succinctly as follows: Given W pairs of observations of α and δ in the time interval $(0,T)$, find a "best estimate" of $\mathbf{y}(t_0)$ and $\mathbf{u}(t)$ so that the state vector $\mathbf{y}(t)$ can then be traced out in state space by numerical integration of Eq. (2). The key difficulty to the solution of this problem is to obtain a suitable expression for $\mathbf{u}(t)$. It is evident that without this complication, i.e., if $\mathbf{u} \equiv 0$, the problem reduces to a standard problem in classical celestial mechanics with many known methods of solution. For example, a quasilinearization approach to this problem was given by Pitkin and Carpenter.[4]

3. Differential Approximation of the Thrust

Although the thrust acceleration $\mathbf{u}(t)$ is completely unknown, we will assume that it is continuous and can be described via the solution to some unknown, nonlinear ordinary differential equation. We further assert that during the time interval $(0,T)$ the thrust can be approximated by a set of functions which are solutions to ordinary linear differential equations. Thus polynomial, exponential, and trigonometric approximations as well as combinations thereof are included automatically. The components of $\mathbf{u}(t)$ are then assumed to satisfy the following differential equation:

$$\begin{aligned} &u_1^M + \sum_{j=1}^{M} \alpha_j u_1^{M-j} = 0 \\ &u_2^N + \sum_{k=1}^{N} \beta_k u_2^{N-k} = 0 \\ &u_3^P + \sum_{l=1}^{P} \gamma_l u_3^{P-l} = 0 \end{aligned} \tag{5}$$

subject to initial conditions

$$\begin{aligned} u_1^j(0) &= d_{j+1};\; j = 0,1,\ldots, M-1 \\ u_2^k(0) &= e_{k+1};\; k = 0,1,\ldots, N-1 \\ u_3^l(0) &= f_{l+1};\; l = 0,1,\ldots, P-1 \end{aligned}$$

where $u_i^j(t)$ represents the jth time derivative of $u_i(t)$. The constants, α_j, β_k, γ_l, d_j, e_k, f_l which completely specify the approximation to $\mathbf{u}(t)$ can ultimately be determined by minimizing a cost function which is obtained from a least squares fit of the solution $\mathbf{y}(t)$ to the observations as given by Eqs. (3) and (4). The details of this procedure will be discussed later.

In order to solve this problem, it is convenient to convert to a system of differential equations in which the constants α_j, β_k, and γ_l become additional state variables. This is accomplished by noting that $\dot{\alpha}_j = \dot{\beta}_k = \dot{\gamma}_l = 0$ and that Eqs. (5) can be easily converted to Mth, Nth, and Pth order systems of first order equations. It is then convenient to unify notation by desigiating the components of the state vector $\mathbf{X}$ as follows

$$\begin{aligned} X_j = u_1^{j-1},\; X_{M+k} &= u_2^{k-1},\; X_{M+N+l} = u_3^{l-1} \\ X_{M+N+P+h} &= y_h;\; h = 1,2,\ldots,6 \\ X_{M+N+P+6+j} &= \alpha_j;\; j = 1,2,\ldots,M \\ X_{2M+N+P+6+k} &= \beta_k;\; k = 1,2,\ldots,N \\ X_{2M+2N+P+6+l} &= \gamma_l;\; l = 1,2,\ldots,P \end{aligned} \tag{6}$$

so that the differential equations become

$$\dot{\mathbf{X}}(t) = \left\{ \begin{array}{l} X_2 \\ X_3 \\ \cdot \\ \cdot \\ \cdot \\ -\sum_{j=1}^{M} X_{M+N+P+6+j} X_{M+1-j} \\ X_{M+2} \\ X_{M+3} \\ \cdot \\ \cdot \\ \cdot \\ -\sum_{k=1}^{N} X_{2M+N+P+6+k} X_{M+N+1-k} \\ X_{M+N+2} \\ X_{M+N+3} \\ \cdot \\ \cdot \\ \cdot \\ -\sum_{l=1}^{P} X_{2M+2N+P+6+l} X_{M+N+P+1-l} \\ X_{M+N+P+4} \\ X_{M+N+P+5} \\ X_{M+N+P+6} \\ -\mu X_{M+N+P+1}/r^3 + X_1 \\ -\mu X_{M+N+P+2}/r^3 + X_{M+1} \\ -\mu X_{M+N+P+3}/r^3 + X_{M+N+1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \right\} \tag{7}$$

where

$$r^2 = X_{I+1}^2 + X_{I+2}^2 + X_{I+3}^2;\; I = M + N + P \tag{8}$$

At this point we have a nonlinear first-order system of Q equations subject to distributed boundary values obtained from Eqs. (3) and (4). The method of solution will now be discussed. We note that M, N and P must be arbitrarily selected, hence, a certain amount of "art" is involved. Generally the lowest possible values that give the desired observation residuals should be selected in order to minimize sensitivity to measurement error and to reduce the order of the system of equations.

4. Quasilinear Method of Solution

Equation (7) may be expressed symbolically as

$$\dot{\mathbf{X}} = \mathbf{f}[\mathbf{X}(t)] \tag{9}$$

A solution which fits the given distributed boundary conditions may be generated by the generalized Newton-Raphson sequence obtained from the truncated Taylor series expansion of $\mathbf{f}$:

$$\dot{\mathbf{X}}^n(t) = \mathbf{f}(\mathbf{X})^{n-1} + \frac{\partial \mathbf{f}(\mathbf{X}^{n-1})}{\partial \mathbf{X}} (\mathbf{X}^n - \mathbf{X}^{n-1})$$

or compactly as

$$\dot{\mathbf{X}}^n(t) = [A^{n-1}]\mathbf{X}^n + \mathbf{B}^{n-1} \tag{10}$$

Here and in what follows the superscript will indicate the iterate number. At each stage of the iteration the quantities with superscript $n - 1$ are presumed to be known so that Eq. (10) characterizes a linear system in $\mathbf{X}^n$, thus the method is commonly called quasilinearization. The solution to Eq. (10) is

$$\mathbf{X}^n(t) = \mathbf{X}_p(t) + \sum_{j=1}^{Q} D_j \mathbf{X}_{H_j}(t) \tag{11}$$

where $\mathbf{X}_p$ denotes a particular solution and $\mathbf{X}_{H_j}$ are Q independent solutions to the homogeneous part of Eq. (10). $\mathbf{X}_p$ is generated by integration from arbitrary initial conditions, $\mathbf{0}$ has been used here for convenience. The $\mathbf{X}_{H_j}$ can likewise be obtained by integrating with independent but otherwise arbitrary initial conditions, e.g., the columns of an identity matrix have been used here to yield a transition matrix or fundamental solution. The D_j are obtained by fitting Eq. (11) to the boundary conditions at the discrete observation times. It should be evident that under these choices of initial conditions that

$$\mathbf{X}^n(0) = \mathbf{D}^n \tag{12}$$

and that upon convergence $\mathbf{D}$ tends to the initial condition vector for the system of equations. Convergence is attained when $\mathbf{X}^n(t)$ and $\mathbf{X}^{n-1}(t)$ differ by less than some preset tolerance.

Presumably the components of $[A^{n-1}]$ should be obtained by operations upon the stored solution, $\mathbf{X}^{n-1}(t)$; however, we have previously shown[4] that the computation is simplified and storage is virtually eliminated with little or no penalty in convergence rate if we generate a new nominal, $\tilde{\mathbf{X}}(t)$, to replace $\mathbf{X}^{n-1}(t)$ at each iteration by an additional integration of Eq. (9) with the latest estimates of $\mathbf{D}$ for the initial conditions. An appropriately revised partial derivative matrix, $[\tilde{A}]$, and $\tilde{\mathbf{B}}$ vector are computed along the new nominal.

We have also shown that accuracy can be greatly improved if, in the latter stages of the iteration, we change to a differential correction algorithm which employs the difference $\Delta\mathbf{X}^n(t) = \mathbf{X}^n(t) - \mathbf{X}^{n-1}(t)$ as the dependent variable in the linearized system. The least significant part of $\Delta\mathbf{X}^n$ which contains the accumulated numerical error is then dropped upon addition to $\mathbf{X}^{n-1}$ with a decided benefit in ultimate accuracy at convergence. The appropriate equations to employ both of these modifications then become

$$\dot{\mathbf{X}}^n(t) = [\tilde{A}]\tilde{\mathbf{X}}^n(t) + \tilde{\mathbf{B}}(t) \tag{13}$$

for the first stages of the iteration and

$$\delta\dot{\mathbf{X}}^n = [\tilde{A}]\delta\mathbf{X}^n \tag{14}$$

where

$$\delta\mathbf{X}^n \triangleq \mathbf{X}^n - \tilde{\mathbf{X}}^{n-1} \tag{15}$$

for the final stages. The form of Eq. (14) suggests differential correction; hence, we so designate this mode of operation. Differential correction requires a much better initial estimate of $\mathbf{X}(0)$ to insure convergence than the regular mode, hence this combination of the two modes is a practical approach to obtain fast convergence and high accuracy from fairly wild initial estimates as has been demonstrated in Ref. 4. Note that little additional coding is required since both modes use the same homogeneous system of equations.

The linearized version of Eq. (7) which corresponds to the general form of Eq. (10) may now be formed by taking indicated partial derivatives on the right hand side. These derivatives are evaluated along the reference path $\tilde{\mathbf{X}}^{n+1}(t)$. As pointed out before, the solution of this linearized system is of the form of Eq. (11) with $\mathbf{D}$ evaluated from the boundary conditions. The precise manner in which this evaluation is made requires some discussion.

First consider a direct solution for which the order of the linearized equations Q is equal to the number of independent bits of information obtained from the observations. From Eq. (3) it can be seen that each observation gives three scalar equations with one additional unknown $\rho(t_i)$ although only two independent quantities, α_i and δ_i, are obtained from the observation at t_i. Thus the number of unknowns is increased by W, the number of observations. The required value of W is obtained by setting the number of observational equations equal to the unknowns, i.e., $3W = Q + W$, to give $W = Q/2$.

It is convenient to specify the boundary conditions in the standard form for a system of linear equations‡

$$[E]\mathbf{D} = \mathbf{C} \tag{16}$$

where $[E]$ is of Q by Q dimension. This is possible if ρ is eliminated by taking a cross product of $\mathbf{L}$ onto Eq. (3) giving

$$\mathbf{L} \times (\mathbf{r} + \mathbf{R}) = 0 \tag{17}$$

Letting $\mathcal{R} = \mathbf{r} + \mathbf{R}$, this can be written in scalar form with any two components being sufficient to provide two independent equations for the boundary conditions at each observation time, e.g.,

$$\begin{gathered} L_1(t_i)\mathcal{R}_2(t_i) - L_2(t_i)\mathcal{R}_1(t_i) = 0 \\ L_1(t_i)\mathcal{R}_3(t_i) - L_3(t_i)\mathcal{R}_1(t_i) = 0 \\ i = 1,2, \ldots, W \end{gathered} \tag{18}$$

Eq. (11) may now be written in the form

$$\mathbf{X}(t) = \mathbf{X}_p(t) + [H(t)]\mathbf{D} \tag{19}$$

where the columns of $[H(t)]$ represent independent solutions of the homogeneous system. Designating $\mathbf{HH}_j{}^T(t)$ as the j^{th} row of $[H(t)]$ allows us to write

$$\begin{gathered} \mathcal{R}_j(t_i) = X_{I+j,p}(t_i) + \mathbf{HH}_{I+j}{}^T(t_i)\mathbf{D} + R_j(t_i), \\ j = 1,2,3 \end{gathered} \tag{20}$$

Then Eq. (18) becomes:

$$\begin{gathered} L_1(t_i)[X_{I+j,p}(t_i) + \mathbf{HH}_{I+j}{}^T(t_i)\mathbf{D} + R_j(t_i)] - \\ L_j(t_i)[X_{I+1,p}(t_i) + \mathbf{HH}_{I+1}{}^T(t_i)\mathbf{D} + R_1(t_i)] = 0, \\ j = 2,3,\ i = 1,2, \ldots W \end{gathered} \tag{21}$$

which can then be easily rewritten in the required form of Eq. (16) if we take the first row of $[E]$ to be

$$L_1(t_1)\mathbf{HH}_{I+2}{}^T(t_1) - L_2(t_1)\mathbf{HH}_{I+1}{}^T(t_1)$$

and the first element of $\mathbf{C}$ to be

$$L_2(t_1)[X_{I+1,p}(t_1) + R_1(t_1)] - L_1(t_1)[X_{I+2,p}(t_1) + R_2(t_1)]$$

etc. The $\mathbf{D}$ vector can then be easily obtained by a standard matrix inversion routine.

Now consider the situation when the number of observations W is greater than $Q/2$ thus providing more observations than are necessary for the Q boundary conditions. Some kind of "fit" or closeness criterion must now be specified. The system of Eq. (21) where now $W > Q/2$ is of the form of Eq. (16) but now $[E]$ is of dimension $2W \times Q$ and $\mathbf{C}$ is of dimension $2W$. We now can specify $\mathbf{D}$ such that the error norm of the difference $[E]\mathbf{D} - \mathbf{C}$ is a minimum in a least squares sense. The well known solution for this criterion is then

$$\mathbf{D} = [E^T E]^{-1}[E]^T\mathbf{C} \tag{22}$$

‡ If range or range-rate data are used, a nonlinear algebraic system arises at this point to complete the solution for $\mathbf{C}$.

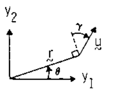

Fig. 2 Thrust direction geometry.

5. Results

In order to see how this proposed method might work in practice, a number of example problems have been explored, two of which are summarized here. The first problem is based upon an orbit generated by the following set of dynamic equations where the trigonometric terms in the last three represent the thrust acceleration.

$$\begin{aligned} \dot{y}_1 &= y_4,\ \dot{y}_2 = y_5,\ \dot{y}_3 = y_6 \\ \dot{y}_4 &= -\mu y_1/r^3 + \sin(\pi/2)t, \\ \dot{y}_5 &= -\mu y_2/r^3 + \cos(\pi/2)t \\ \dot{y}_6 &= -\mu y_3/r^3 + \cos(\pi/2)t \end{aligned} \tag{23}$$

Observations for this system were generated by numerically integrating the system from the initial conditions:

$$y(0)^T = [-1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]$$

The trajectory was then reduced to observations on the unit vector $\mathbf{L}$ by inversion of the geometric Eq. (3) to

$$\mathbf{L} = (\mathbf{r} + \mathbf{R})/(\rho \cdot \rho)^{1/2} \tag{24}$$

giving the twelve observations of Table 1 as made from three observation platforms. The positions of the force center with respect to these platforms were taken as follows:

Station no. 1) $\mathbf{R} = 0.5\mathbf{i} + 0.5\mathbf{j} - 0.5\mathbf{k}$

2) $\mathbf{R} = 0.5\mathbf{i} + 0.3\mathbf{j} - 0.4\mathbf{k}$

3) $\mathbf{R} = 0.5\mathbf{i} - 0.4\mathbf{j} - 0.4\mathbf{k}$

This choice of stations assures that the observer is not in the orbit plane but is otherwise arbitrary.

The problem then is to use the proposed orbit determination scheme to reproduce the initial conditions and the thrust program which appears as the last terms of the last three of Eq. (23). Since the "unknown" thrust here can be characterized by the solution of a linear differential equation, it follows that one should be able to model the thrust program precisely and obtain the exact solution for the augmented state vector with an assumed second-order system such that $M = N = P = 2$. This solution is

$$\mathbf{X}^T(0) = [0\ (\pi/2)\ 1\ 0\ 1\ 0\ -1\ 0\ 0\ 0\ 1\ 0\ 0\ (\pi/2)^2\ 0\ (\pi/2)^2] \tag{25}$$

An integration stepsize of $\Delta t = 0.01$ was used with a fourth-order Runge-Kutta integration routine in the program.

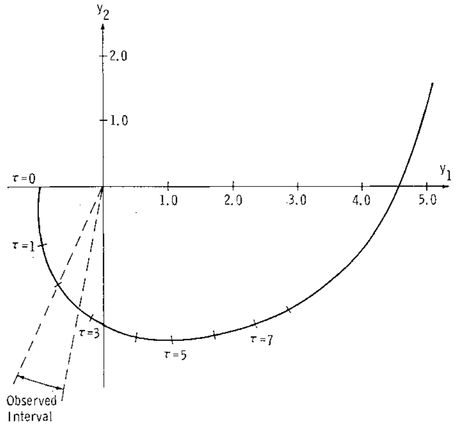

Fig. 3 Section of logarithmic spiral.

Table 1 Observations of the trajectory with trigonometric thrust functions

	Observations			
Time	L_1	L_2	L_3	Station
0	−0.5773502691	0.5773502691	−0.5773502691	1
0.020	−0.5694969088	0.5927426106	−0.5694993137	1
0.040	−0.5613276129	0.6081127756	−0.5613467404	1
0.060	−0.6794097189	0.4934008714	−0.5431003718	2
0.080	−0.6691558037	0.5161586823	−0.5346126868	2
0.100	−0.6583797307	0.5387130733	−0.5256656301	2
0.120	−0.7171680140	−0.3978225115	−0.5721951493	3
0.140	−0.7258712128	−0.3717750385	−0.5787005296	3
0.160	−0.7343184194	−0.3443423925	−0.5849827138	3
0.180	−0.7424408650	−0.3154283342	−0.5910046766	3
0.200	−0.7501576401	−0.2849332680	−0.5967214993	3
0.220	−0.7573739465	−0.2527556625	−0.6020791311	3

The results of the experiment are summarized in Table 2 which shows the convergence of the state vector $\mathbf{X}(0)$. The zeroth iteration represents the initial guess which must be made for all components of $\mathbf{X}(0)$. For the first three iterations the program operates in the regular mode, i.e., using Eq. (13), and with a direct solution for the first 9 observations. Note that during these iterations the constants α_1, α_2, β_1, β_2, γ_1, and γ_2 have been forced to retain their initially estimated values. It has been found that the scheme of forcing these constants to remain at these estimated values until the other components converge to an intermediate value significantly increase the region of convergence. On the fourth iteration this constraint is relaxed and the program shifts to the differential correction mode of Eq. (14). The state vector continues to converge and then on the 11th iteration the program switches to a least squares solution for all 12 observations. Thus, by the 12th iteration, all of the components of the augmented state vector have been determined to at least five decimal places as can be seen by comparison of the results to Eq. (25). The trajectory predicted by this solution compares to the generated solution to at least 10 decimal places and the thrust programs to at least 7 decimal places over the observation interval, indicating very good precision.

The second example problem considered here is a trajectory which follows a logarithmic spiral. The spiral was selected because both the trajectory and thrust program can be expressed in closed form, thus eliminating the need of having to integrate the dynamical equations to generate the observations. Knowing the analytic solution makes it possible to determine the absolute accuracy of the orbit determination program for this case. The trajectory is given by the equation

$$r = r_0 e^{(\theta - \theta_0)\tan\gamma} \tag{26}$$

where r_0 is the initial radius when $\theta = \theta_0$. This trajectory in the y_1, y_2 plane will be generated by a thrust program which satisfies

$$\begin{aligned} u_1(\theta) &= (\mu \sin\gamma/2r^2)\sin(\gamma - \theta), \\ u_2(\theta) &= (\mu \sin\gamma/2r^2)\cos(\gamma - \theta) \\ u_3(\theta) &= 0 \end{aligned} \tag{27}$$

where γ, the thrust direction angle, is a constant for the trajectory and is shown in Fig. 2. Bacon[5] has shown that it is possible to relate time to the trajectory by the following equation which gives the time τ to reach a given radius r from the initial radius r_0 as

$$\tau = 2(r^{3/2} - r_0^{3/2})/3(\mu)^{1/2}\sin\gamma \tag{28}$$

Equations (26–28) were used to generate the trajectory and thrust programs shown in Figs. 3 and 4 with $\mu = 1$, $\gamma = 0.45$ radians, and $r_0 = 1$ when $\theta_0 = \pi$ or $y_1(0) = -1$, $y_2(0) = 0$, $y_3(0) = 0$. Table 3 summarizes the observations of the trajectory over the time interval [2.0, 2.6] from stations located on the Earth's surface. The positions of the force center with

Table 2 Convergence of the state vector

Problem 1		$M = N = P = 2$		Stepsize = 0.01		No. of observations = 12	
Mode	Iteration	d_1	d_2	e_1	e_2	f_1	f_2
	0	0.5000000000	0.5000000000	0.5000000000	0.3000000000	0.6000000000	0.3000000000
1	1	1.4400373355	−1.0693734368	−1.6764710243	−7.0924698596	0.8863658374	3.7531823248
1	2	0.0015173486	1.5190954203	0.9985752686	0.0471382195	1.0015226794	−0.0520635673
1	3	−0.0000825713	1.5733554710	1.0000797078	−0.0024776530	0.9999149771	0.0027863825
2	4	−0.0000826128	1.5733567911	1.0000797459	−0.0024788291	0.9999149350	0.0027877306
2	10	0.0000000195	1.5707957126	0.9999999838	0.0000004071	1.0000000194	−0.0000005932
2	11[a]	0.0000000015	1.5707962896	0.9999999989	0.0000000256	−0.0000000016	−0.0000000353
2	12	0.0000000015	1.5707962897	0.9999999989	0.0000000256	1.0000000016	−0.0000000352
		$y_1(0)$	$y_2(0)$	$y_3(0)$	$y_4(0)$	$y_5(0)$	$y_6(0)$
	0	−0.5000000000	0.2000000000	0.0000000000	0.3000000000	0.7000000000	0.0000000000
	1	−1.0000438058	0.0000438058	−0.0000438058	0.0025152329	0.9975577746	0.0025189128
	2	−0.9999994285	−0.0000005715	0.0000005715	−0.0000331495	1.0000321739	−0.0000331699
	3	−1.0000000311	0.0000000311	−0.0000000311	−0.0000018389	0.9999982008	0.0000018513
	4	−1.0000000311	0.0000000311	−0.0000000311	0.0000018398	0.9999981997	0.0000018522
	10	−1.0000000000	−0.0000000000	0.0000000000	−0.0000000005	1.0000000005	−0.0000000005
	11[a]	−1.0000000000	−0.0000000000	0.0000000000	−0.0000000001	1.0000000000	−0.0000000000
	12	−1.0000000000	−0.0000000000	0.0000000000	−0.0000000001	1.0000000000	−0.0000000000
		α_1	α_2	β_1	β_2	γ_1	γ_2
	0	0.5000000000	1.0000000000	0.5000000000	1.0000000000	0.5000000000	1.0000000000
	1	0.5000000000	1.0000000000	0.5000000000	1.0000000000	0.5000000000	1.0000000000
	2	0.5000000000	1.0000000000	0.5000000000	1.0000000000	0.5000000000	1.0000000000
	3	0.5000000000	1.0000000000	0.5000000000	1.0000000000	0.5000000000	1.0000000000
	4	0.0303458778	2.0273669202	−1.1098830141	2.4168652649	−0.0846958684	2.5271057385
	10	−0.0000088994	2.4675101303	0.0000120109	2.4674070360	−0.0000618380	2.4673880970
	11[a]	−0.0000003836	2.4674040398	0.0000007656	2.4674014252	−0.0000016919	2.4674005448
	12	−0.0000003838	2.4674040459	0.0000007658	2.4674014253	−0.0000016947	2.4674005447

[a] Change to least squares

respect to these stations are given by

Station no. 1) $\mathbf{R} = +0.4\mathbf{i} + 0.75\mathbf{j} - 0.5\mathbf{k}$

2) $\mathbf{R} = -0.4\mathbf{i} + 0.75\mathbf{j} - 0.5\mathbf{k}$

3) $\mathbf{R} = +0.0\mathbf{i} + 0.80\mathbf{j} + 0.6\mathbf{k}$

Since the thrust programs given in Eq. (27) cannot be written as explicit functions of time, it is impossible to write an analytic solution for the augmented state vector as was done for the first problem. For the experiment being discussed, the first two components of thrust were approximated by second-order systems while the third was estimated by a constant, thus $M = N = 2$, $P = 0$. Once again a stepsize of $\Delta t = 0.01$ was used in the integration routine. The results for this experiment are shown in Table 4. It should be noted on the second iteration that the first eleven components of the state vector converged to at least four decimal places of the final solution even though the constants α_1, α_2, β_1, β_2 were forced to remain at their initially guessed values. It was found that if these constants were allowed to take on their predicted values early in the process that the iteration quickly diverged. The trajectory produced by the 8th iteration agrees with the generated trajectory to at least nine decimal places while the thrust programs compare to at least six decimal places over the observation interval. The solution was extrapolated forward by one observation period to $\tau = 3.28$. It was found that the predicted trajectory was still accurate to six decimal places while the thrust programs were reduced to four decimal accuracy. This suggests that the method holds limited promise for orbit prediction as well as orbit determination.

Table 3 Observations for the logarithmic spiral

	Observations			
Time	L_1	L_2	L_3	Station
2.0194122623	−0.2891769223	−0.8264894955	−0.4830029206	1
2.0712628631	−0.2599053369	−0.8407240713	−0.4750076333	1
2.1238703357	−0.2306432303	−0.8536762642	−0.4669481085	1
2.1772457283	−0.6907198916	−0.6388702491	−0.3387489277	2
2.2314002503	−0.6718779797	−0.6585716313	−0.3389150140	2
2.2863452748	−0.6522922869	−0.6779321109	−0.3390026921	2
2.3420923408	−0.4207335947	−0.7732350459	0.4744373572	3
2.3986531560	−0.3935309611	−0.7894480640	0.4710680807	3
2.4560395988	−0.3658566948	−0.8047304863	0.4675015755	3
2.5142637209	−0.3377619198	−0.8190588018	0.4637451505	3
2.5733377502	−0.3092986802	−0.8324133189	0.4598069081	3
2.6332740928	−0.2805195562	−0.8447782191	0.4556956648	3

6. Summary and Recommendations for Future Studies

The analytic technique of differential approximation in conjunction with the modified method of quasilinearization has been used to estimate the time history of an unknown vehicle's position, velocity, and thrust program from angular observations taken at discrete times. This was accomplished by approximating the vehicle's unknown thrust program in such a way that the complete system dynamics could be specified by ordinary differential equations. A modified method of quasilinearization was then used to solve the resulting multipoint boundary value problem. The feasibility of the method was established by applying the algorithm to a logarithmic spiral orbit which was first generated by a known analytic thrust program. Observations generated from the known solution were then used as the input to the orbit determination algorithm. The algorithm showed excellent

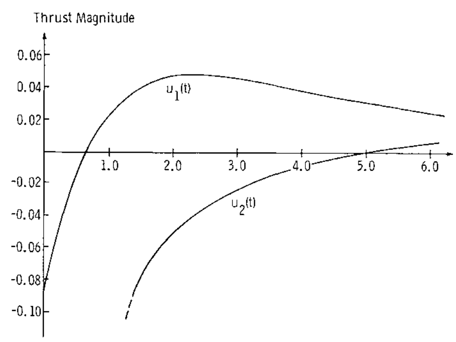

Fig. 4 Thrust program for logarithmic spiral.

Table 4 Convergence of the state vector for the logarithmic spiral

Problem 2		$M = N = 2, P = 0$		Stepsize = 0.1		No. of observations = 12	
Mode	Iteration	d_1	d_2	e_1	e_2	f_1	$y_1(0)$
	0	0.5000000000	−0.2000000000	0.0010000000	−0.1000000000	0.2000000000	−1.0000000000
1	1	0.0995406929	−0.7866146399	−0.1983495688	0.5282798489	−0.0426743478	−0.6993353352
1	2	0.0462297505	0.0033459834	−0.0537755773	0.0381398848	0.0000007764	−0.6993531685
1	3	0.0462220620	0.0035580875	−0.0537762225	0.0381281090	−0.0000000240	−0.6993531820
1	4	0.0462220620	0.0035580873	−0.0537762225	0.0381281092	−0.0000000240	−0.6993531820
2	5	0.0462228194	0.0035329993	−0.0537764935	0.0381450326	0.0000000402	0.6993531814
2	6	0.0462225905	0.0035400395	−0.0537765234	0.0381448555	0.0000000132	−0.6993531818
2	7[a]	0.0462227296	0.0035372634	−0.0537766290	0.0381452776	−0.0000000105	−0.6993531820
2	8	0.0462227297	0.0035372619	−0.0537766286	0.0381452714	−0.0000000105	−0.6993531820
		$y_2(0)$	$y_3(0)$	$y_4(0)$	$y_5(0)$	$y_6(0)$	α_1
	0	−1.0000000000	0.2000000000	1.0000000000	−1.0000000000	0.3000000000	0.5000000000
	1	−1.6055320071	0.0000298088	0.4922117232	−0.5722807601	0.0004358049	0.5000000000
	2	−1.6055739761	0.0000000244	0.4925604206	−0.5730593947	−0.0000002230	0.5000000000
	3	−1.6055740146	−0.0000000001	0.4925606762	−0.5730590358	0.0000000030	0.5000000000
	4	−1.6055740146	−0.0000000001	0.4925606762	−0.5730590358	0.0000000030	0.9191045175
	5	−1.6055740129	0.0000000008	0.4925606587	−0.5730590514	−0.0000000091	1.4066445932
	6	−1.6055740139	0.0000000003	0.4925606656	−0.5730590417	−0.0000000030	1.4904159220
	7[a]	−1.6055740145	−0.0000000001	0.4925606648	−0.5730590303	0.0000000017	1.4848433811
	8	−1.6055740145	−0.0000000001	0.4925606648	−0.5730590303	0.0000000017	1.4848381217
		α_2	β_1	β_2			
	0	1.0000000000	0.5000000000	1.0000000000			
	1	1.0000000000	0.5000000000	1.0000000000			
	2	1.0000000000	0.5000000000	1.0000000000			
	3	1.0000000000	0.5000000000	1.0000000000			
	4	0.3062834452	0.6186190760	0.0072413358			
	5	0.2589382012	0.3736868283	−0.1752886607			
	6	0.2553982585	0.3497473195	−0.1923070786			
	7[a]	0.2553215480	0.3555599575	−0.1881484830			
	8	0.2553215290	0.3556036743	−0.1881165070			

[a] Change to least squares

precision in these examples as the numerical solution was able to regenerate the known solution to at least nine significant figures over the period of observation.

Only self-generated orbits were considered here. It is therefore important to apply the method to real observational data to determine the effect of measurement noise on the convergence and precision of the algorithm. Also, if this method is to be used as a successful tracking technique for powered vehicles, it must be adapted to sequential data filtering techniques such that the solution is sequentually updated as new observational data are recorded without having to resort to a full iteration process.

This paper has considered the problem of estimating the unknown thrust program of a space vehicle, yet the techniques used can easily be applied to determining any unknown perturbing force influencing the motion of a vehicle or a celestial body. For example, the drag forces acting on a reentry vehicle might be determined from tracking information. Once a solution is estimated by the algorithm, the drag forces could be expressed as a function of altitude and might be used to infer atmospheric parameters such as density variations.

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